SOME PROBLEMS OF OPTIMAL CONTROL WITH A SMALL PARAMETER

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Optimal control systems containing a small parameter which can be called weakly controlled systems are considered. A procedure for the approximate solutions of problems of this class is described. A variational problem on the attainment of maximum gliding range by a craft with aerodynamic controls in the atmosphere is solved as an example. The results obtained are in good agreement with the exact numerical solution.

1. Formulation of the problem. Let the controlled process be described by a system of differential equations with the initial conditions

$$dx / dt = f(x, t, u), \qquad x(t_0) = a \tag{1.1}$$

Here t is the time, $x = (x_1, \ldots, x_n)$ is the n-dimensional phase coordinate vector, $u = (u_1, \ldots, u_m)$ is the m-dimensional vector of the controlling functions, $f = (f_1, \ldots, f_n)$ is a given n-dimensional vector function, t_0 is the initial instant, and a is the vector of the initial phase state. The conditions at the end of the process and the functional J to be minimized are given in the form

$$h(x(T), T) = 0, \quad q(x(T), T) = 0, \quad J = F(x(T), T) \tag{1.2}$$

Here $h(x, t)$ and $F(x, t)$ are given scalar functions; $q(x, t) = (q_1, \ldots, q_r)$ is a given r-dimensional vector function, $0 \leqslant r < n - 1$. The first Eq. of (1.2) is the condition which defines the instant T of termination of the process. We assume that the function h depends monotonously on t (over some time interval) for the permissible trajectories $x(t)$, and that the condition $h = 0$ defines a unique instant T for each permissible trajectory. The second (vector) equation of (1.2) imposes additional boundary conditions at the instant T (if $r = 0$, these conditions are lacking). All these conditions are assumed to be independent and noncontradictory.

Our problem consists in determining the optimal control $u(t)$ and the corresponding optimal trajectory $x(t)$ which for $t_0 \leqslant t \leqslant T$ satisfy Eqs. and conditions (1.1) and (1.2) as well as the restrictions on the control $u(t) \in U$, and which minimize the functional J. Here U is a given closed set in m-dimensional space.

Let us introduce the additional phase coordinates x_0 and x_{n+1} subject to the equations and initial conditions

$$dx_0/dt = f_0, \quad dx_{n+1}/dt = 1, \quad x_0(t_0) = 0, \quad x_{n+1}(t_0) = t_0$$

$$f_0 = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \left(\frac{\partial F}{\partial x}, f\right) \tag{1.3}$$

Here and below $\partial/\partial x$ is the gradient operator over the phase coordinates x; d/dt is the total derivative along the trajectories of system (1.1); the parentheses denote scalar products.

It is clear that $x_{n+1} \equiv t$, so that the argument t of the functions f, f_0, h, q, and F can be replaced by x_{n+1}, which makes the system self-contained. Functional (1.2) then takes the form $J = x_0(T)$.

Let us apply the maximum principle [1] to the problem just formulated. Introducing the vector of conjugate variables $\psi(t) = (\psi_1, \ldots, \psi_n)$ and the conjugate variables $\psi_{n+1}(t)$ and $\psi_0(t)$, we assume, as usual, that $\psi_0 \equiv -1$. The Hamiltonian H' and the conjugate equations for systems (1.1) and (1.3) become

$$H' = (\psi, f) + \psi_{n+1} - f_0 = (\psi - \partial F / \partial x, f) + \psi_{n+1} - \partial F / \partial t \qquad (1.4)$$

$$\frac{d\psi_k}{dt} = -\frac{\partial H'}{\partial x_k} = -\left(\psi - \frac{\partial F}{\partial x}, \frac{\partial f}{\partial x_k}\right) + \left[\frac{\partial^2 F}{\partial t\, \partial x_k} + \left(\frac{\partial}{\partial x_k}\frac{\partial F}{\partial x}, f\right)\right] \quad (k = 1, \ldots, n)$$

With allowance for boundary conditions (1.2) (the instant of termination of the process has not been fixed), we can write the transversality conditions in the form

$$\psi = \lambda \frac{\partial h}{\partial x} + \sum_{i=1}^{r} \lambda_i \frac{\partial q_i}{\partial x}, \qquad \psi_{n+1} = \lambda \frac{\partial h}{\partial t} + \sum_{i=1}^{r} \lambda_i \frac{\partial q_i}{\partial t}, \qquad H' = 0 \qquad (1.5)$$

Here λ and λ_i are constant parameters. Let us substitute Conditions (1.5) into Eq. (1.4) for H' and then solve the latter for λ;

$$\lambda = \left(\frac{dF}{dt} - \sum_{i=1}^{r} \lambda_i \frac{dq_i}{dt}\right)\left(\frac{dh}{dt}\right)^{-1} \qquad \text{for} \quad t = T \qquad (1.6)$$

The total derivatives have the same meaning here as in Eq. (1.3). We now introduce the notation

$$p = \psi - \partial F/\partial x, \; H = (p, f) = H' - \psi_{n+1} + \partial F / \partial t, \; p = (p_1, \ldots, p_n) \quad (1.7)$$

The expression in square brackets in (1.4) is equal to $d(\partial F/\partial x_k)/dt$. Eqs. (1.4) and conditions (1.5) with allowance for (1.7) can be written as

$$\frac{dp_k}{dt} = -\left(p, \frac{\partial f}{\partial x_k}\right) = -\frac{\partial H}{\partial x_k}, \qquad H = (p, f)$$

$$p = \lambda \frac{\partial h}{\partial x} + \sum_{i=1}^{r} \lambda_i \frac{\partial q_i}{\partial x} - \frac{\partial F}{\partial x} \qquad \text{for} \quad t = T \qquad (1.8)$$

By applying the maximum principle we have reduced the optimal control problem to a boundary value problem for the two n-dimensional vector functions $x(t)$ and $p(t)$. The control $u(t)$ can be found from the supremum condition for the function H' with respect to u. This is equivalent to the supremum of the function H from (1.8), i.e. to

$$H(p(t), x(t), t, u(t)) = \sup_{u \in U} H(p(t), x(t), t, u) \qquad (1.9)$$

The system of equations of the boundary value problem consists of Eqs. (1.1) and (1.8), and the boundary conditions of Eqs. (1.1), (1.2) and (1.8). The control u can be eliminated by means of Eq. (1.9).

The parameter λ is defined by Eq. (1.6); the instant T and the parameters λ_i are unknown and must be determined in the course of solving the problem.

Let us expand the functions f, h, q, and F and the vector a in series in the small parameter ε,

$$f = f^0(x, t) + \varepsilon f^1(x, t, u) + \ldots, \qquad h = h^0(x, t) + \varepsilon h^1(x, t) + \ldots$$
$$q = q^0(x, t) + \varepsilon q^1(x, t) + \ldots, \qquad F = F^0(x, t) + \varepsilon F^1(x, t) + \ldots$$
$$a = a^0 + \varepsilon a^1 + \ldots \qquad (\varepsilon \ll 1) \qquad (1.10)$$

The superscripts in all cases denote the number of terms in the expansions; the subscripts denote the number of vector components. Since the function f does not depend on u

for $\varepsilon = 0$, system (1.1) is uncontrolled when $\varepsilon = 0$. We will assume that its general solution is known. It is natural to call system (1.1) for $0 < \varepsilon \ll 1$ a "weakly controlled" system. In the next section we shall construct an approximate solution of the above optimal control problem for a weakly controlled system.

If the function f^0 depends on u, then the system does not degenerate into an uncontrolled system for $\varepsilon = 0$ and there generally exists an optimal control of the zeroth approximation. Expansion in the small parameter serves merely to refine this control. The case considered in the present paper (where the system is uncontrolled for $\varepsilon = 0$) is interesting in that the control in the zeroth approximation cannot be determined in principle. An intermediate case is also possible: this is where the function f^0 depends only on certain components of the vector of controlling functions.

We note also that if the set U depends on x, t, and ε, then in a number of cases it can be transformed into a constant set by simple transformation in the control space. The set U defined by the inequality $|u| \leqslant C(x, t, \varepsilon)$ (where C is a known function), for example, can be transformed into the set $|u'| \leqslant 1$ by means of the transformation $u = Cu'$. From now on we shall assume that the set U is constant.

Neither the problems involved in constructing strict estimates of the error of the approximate solution nor the existence and uniqueness of this solution will be considered in the present paper.

2. The approximate solution. We shall attempt to find the solution of the above problem and the functional J for $\varepsilon \ll 1$ in the form

$$x = x^0(t) + \varepsilon x^1(t) + \ldots, \quad p = p^0(t) + \varepsilon p^1(t) + \ldots, \quad T = T^0 + \varepsilon T^1 + \ldots$$
$$\lambda = \lambda^0 + \varepsilon\lambda^1 + \ldots, \quad \lambda_i = \lambda_i^0 + \varepsilon\lambda_i^1 + \ldots, \quad J = J^0 + \varepsilon J^1 + \ldots \quad (i=1,\ldots r). \tag{2.1}$$

Substituting Eqs. (2.1) and (1.10) into Eqs. (1.1), (1.2), (1.8), and (1.6) we expand the resulting expressions in series in ε and equate the coefficients of ε^0 and ε^1. In the zeroth approximation we have

$$dx^0/dt = f^0(x^0, t), \quad x^0(t_0) = a^0, \quad h^0(x^0(T^0), T^0) = 0, \quad q^0(x^0(T^0), T^0) = 0$$
$$J^0 = F^0(x^0(T^0), T^0) \tag{2.2}$$
$$\frac{dp_k^0}{dt} = -\left(p^0, \frac{\partial f^0(x^0(t), t)}{\partial x_k}\right), \qquad p^0 = \lambda^0\frac{\partial h^0}{\partial x} + \sum_{i=1}^{r}\lambda_i^0\frac{\partial q_i^0}{\partial x} - \frac{\partial F^0}{\partial x}$$
$$\lambda^0 = \left\{\frac{\partial F^0}{\partial t} + \left(\frac{\partial F^0}{\partial x}, f^0\right) - \sum_{i=1}^{r}\lambda_i^0\left[\frac{\partial q_i^0}{\partial t} + \left(\frac{\partial q_i^0}{\partial x}, f^0\right)\right]\right\}\left[\frac{\partial h^0}{\partial t} + \left(\frac{\partial h^0}{\partial x}, f^0\right)\right]^{-1}$$
$$\text{for } t = T^0 \ (k = 1, \ldots, n)$$

We also write out the equations of the first approximation for Eqs. (1.1) and (1.2) (we make use of relations (2.2) obtained above in constructing these equations),

$$\frac{dx_k^1}{dt} = \left(\frac{\partial f_k^0(x^0(t), t)}{\partial x}, x^1\right) + f^1(x^0(t), t, u(t)), \quad x^1(t_0) = a^1$$
$$\left[\frac{\partial h^0}{\partial t} + \left(\frac{\partial h^0}{\partial x}, f^0\right)\right]T^1 + \left(\frac{\partial h^0}{\partial x}, x^1(T^0)\right) + h^1 = 0$$
$$\left[\frac{\partial q_i^0}{\partial t} + \left(\frac{\partial q_i^0}{\partial x}, f^0\right)\right]T^1 + \left(\frac{\partial q_i^0}{\partial x}, x^1(T^0)\right) + q_i^1 = 0 \tag{2.3}$$
$$J^1 = \left[\frac{\partial F^0}{\partial t} + \left(\frac{\partial F^0}{\partial x}, f^0\right)\right]T^1 + \left(\frac{\partial F^0}{\partial x}, x^1(T^0)\right) + F^1$$
$$(i = 1, \ldots, r)$$

In the last three Eqs. of (2.3) all the functions of x and t are taken for the values $x = x^0(T^0)$, $t = T^0$.

Now let us analyse Eqs. (2.2) and (2.3). We assume the general solution for the zeroth-approximation system $dx/dt = f^0(x, t)$ of (2.2) to be known and to be given in the form

$$x = \varphi(t, c), \quad \varphi = (\varphi_1, \ldots, \varphi_n), \quad c = (c_1, \ldots, c_n) \tag{2.4}$$

Here φ is a vector function and c is a vector of arbitrary constants. Solving Eqs. (2.4) for the constants c, we obtain

$$g(x, t) = c, \qquad (g = g_1, \ldots, g_n) \tag{2.5}$$

The functions g_k are the independent first integrals of the zeroth-approximation system.

For the trajectory in the zeroth approximation we have Cauchy problem (2.2) whose solution can be expressed in terms of the functions φ and g introduced by way of Eqs. (2.4) and (2.5),

$$x^0(t) = \varphi(t, c), \qquad c = g(a^0, t) \tag{2.6}$$

The instant T^0 of termination of the process and the functional J^0 in this approximation are given by the third and fifth Eqs. of (2.2). We shall assume that the fourth Eq. of (2.2), i.e. the boundary conditions $q = 0$, are fulfilled automatically in this approximation. This equation can be considered as an additional condition imposed on the function $q^0(x, t)$.

Let us introduce the following $n \times n$ matrices:

$$\Phi(t, c) = \left\| \frac{\partial \varphi_i}{\partial c_j} \right\|, \qquad G(t, c) = \left\| \frac{\partial g_i}{\partial x_j} \right\| \quad \text{for} \quad x = \varphi(t, c) \tag{2.7}$$

Eqs. (2.4) and (2.5) define transformations which transform the vector c into x, and vice-versa. Matrices (2.7) which are the Jacobi matrices for these mutually inverse transformations, are related to each other by the expression $\Phi = G^{-1}$. The rank of both matrices is n.

The function x^1 satisfies linear homogeneous system (2.3). The corresponding homogeneous system is a system in variations for zeroth-approximation system (2.2) satisfied by x^0. As we know from the theory of differential equations, the matrix Φ of (2.7) is the fundamental matrix for the system in variations. Making use of this fact, let us write out the general solution of inhomogeneous system (2.3) by the method of variation of arbitrary constants [2],

$$x^1 = \Phi(t, c)\, b + \Phi(t, c) \int_{t_0}^{t} \Phi^{-1}(\tau, c)\, f^1(x^0(\tau), \tau, u(\tau))\, d\tau$$

Determining the vector b of arbitrary constants with the aid of initial condition (2.3) and making use of the Eq. $\Phi^{-1} = G$, we obtain

$$x^1(t) = \Phi(t, c)\, G(t_0, c)\, a^1 + \Phi(t, c) \int_{t_0}^{t} G(\tau, c)\, f^1(x^0(\tau), \tau, u(\tau))\, d\tau \tag{2.8}$$

Let us also express T^1 from the third Eq. of (2.3) and then substitute it into the fourth Eq. of (2.3),

$$\left[\left(\frac{\partial q_i^0}{\partial x}, x^1(T^0)\right) + q_i^1\right]\left[\frac{\partial h^0}{\partial t} + \left(\frac{\partial h^0}{\partial x}, f^0\right)\right] = \left[\left(\frac{\partial h^0}{\partial x}, x^1(T^0)\right) + h^1\right] \times$$
$$\times \left[\frac{\partial q_i^0}{\partial t} + \left(\frac{\partial q_i^0}{\partial x}, f^0\right)\right] \quad (i = 1, \ldots, r) \tag{2.9}$$

As is evident from (2.2), the vector p^0 satisfies the linear homogeneous system conjugate to the above system in variations. But then, as we know [2], the fundamental matrices for this system is $(\Phi^{-1})' = G'$, where the prime denotes the transposed matrix. Hence, the general solution of system (2.2) for p^0 (in vector and scalar notation) is of the form

$$p^0 = G'(t, c)\, s, \quad p_k^0 = \sum_{i=1}^{n} \frac{\partial g_i}{\partial x_k} s_i, \quad s = (s_1, \ldots, s_n) \quad (k = 1, \ldots, n) \tag{2.10}$$

Here s is a vector of arbitrary constants. Substituting solution (2.10) into condition (2.2) for p^0 and taking account of the Eq. $(G')^{-1} = \Phi'$, we obtain

$$s = \Phi'(T^0, c)\left[\lambda^0 \frac{\partial h^0}{\partial x} + \sum_{i=1}^{r} \lambda_i^0 \frac{\partial q_i^0}{\partial x} - \frac{\partial F^0}{\partial x}\right] \quad \text{for } t = T^0 \qquad (2.11)$$

Now let us determine the control in the first approximation (the system is uncontrolled in the zeroth approximation). Substituting expansions (1.10) and (2.1) into the function H from (1.8), we expand this function in a series in ε,

$$H = (p, f) = (p^0, f^0(x^0, t)) + \varepsilon\left[\left(p^0, \sum_{i=1}^{n} \frac{\partial f^0}{\partial x_i} x_i^1\right) + \left(p^1, f^0(x^0, t)\right) + \right.$$
$$\left. + (p^0, f^1(x^0, t, u))\right] + \ldots$$

The three dots denote terms of order higher than the first. Of the terms written out above only the last depends on u. Hence, the determination of the maximum of H with respect to u reduces in the first approximation to the maximization of this last term, i.e. to

$$(p^0(t), f^1(x^0(t), t, u(t))) = \sup\, (p^0(t), f^1(x^0(t), t, u))\ (u \in U) \qquad (2.12)$$

The control $u(t)$ defined by relation (2.12) need not lie close to the control optimal in the metric sense in the space of C (i.e. with respect to the maximum of the difference modulus). However, this control will be approximately optimal in the sense of the functional to be minimized. In fact, the familiar formulas for the first variation of the functional [3] imply that the functionals for two different controls differ by an amount of the same order as the functions H for these controls. But if condition (2.12) is fulfilled, the function H for the control $u(t)$ will differ from the maximum of the function H attained in choosing the optimal control by an amount of the same order as the rejected terms, i.e. by an amount on the order of ε^2. The difference with respect to the functional between the approximate and optimal controls will be of the same order of magnitude. The difference in norm between these controls in the space L_2, i.e. the mean-square error, will usually be on the order of ε.

We note that in accordance with (2.12) the control $u(t)$ depends only on the solutions $x^0(t)$ and $p^0(t)$ of the zeroth approximation. Substituting in solution (2.10), we can rewrite condition (2.12) as

$$(G's, f^1) = \sum_{i,j=1}^{n} \frac{\partial g_i}{\partial x_j} s_i f_j^1(x^0(t), t, u) \to \sup \quad \text{with respect to } u \in U \qquad (2.13)$$

The resulting relations enable us to obtain an approximate solution of the optimal control problem under investigation. Here the trajectory $x(t)$ as well as the instant T and the functional J will be determined in the first approximation (with allowance for two terms in expansions (2.1), i.e. to within $\sim\varepsilon^2$); the conjugate variables $p(t)$ and the constants λ and λ_i will be found in the zeroth approximation. The subsequent terms of expansions (2.1) are too small to be of much interest.

Finding the approximate solution involves the following steps:

1. Finding the general solution of the zeroth-approximation system, i.e. finding the functions φ, g of (2.4) and (2.5) and the matrices Φ, G of (2.7).

2. In the zeroth approximation the trajectory $x^0(t)$ is defined by Eqs. (2.6). The instant T^0 and the functional J^0 are defined by the third and fifth Eqs. of (2.2). The fourth Eq. of (2.2) is assumed to be fulfilled by hypothesis.

3. The function $p^0(t)$ is defined by Eqs. (2.10) and the vector s by Eq. (2.11) into which we must substitute λ^0 from (2.2). The right sides of Eqs. (2.11) and (2.2) must be taken for $x = x^0(T^0)$, $t = T^0$. Thus, Eqs. (2.10), (2.11), and (2.2) define the function $p^0(t)$ to within r arbitrary constants λ_i^0 which will be determined below.

4. Substituting $x^0(t)$ and $p^0(t)$ into condition (2.12) or (2.13) and computing the supremum with respect to u, we obtain the control $u(t)$ also to within r unknown constants λ_i^0.

5. We substitute $x^0(t)$ and $u(t)$ into Eq. (2.8) and find $x^1(t)$, and in particular $x^1(T^0)$, to within the same constants.

6. We then substitute $t = T^0$, $x = x^0(T^0)$ and the resulting value into relations (2.9). This yields r algebraic (generally nonlinear) equations for the constants λ_i^0 appearing in $x^1(T^0)$. Solving these equations (we assume that a solution exists), we find the constants λ_i^0. The functions $p^0(t)$, $u(t)$, $x^1(t)$ and the constant λ^0 determined in Steps 3 to 5 have now been determined completely.

7. The corrections T^1 for the instant of process termination and J^1 for the functional can be found consecutively from the third and fifth Eqs. of (2.3) by substituting in them the already known values of $x = x^0(T^0)$, $t = T^0$, and $x^1(T^0)$.

Let us consider the solution of our problem for the case where the boundary conditions $q = 0$ (except the condition $h = 0$ which serves to define the instant of termination of the process) are lacking at the end of the process. In this case the dimensionality r of the vector q of (1.2) is zero, so that the equations of Sections 1 and 2 lack the terms containing the functions q_i, q_i^0 and the constants λ_i, λ_i^0. Relations (2.9) must also be omitted. Approximate solution of the problem is simpler in this case, since its most complicated stage, i.e. the solution of the system of algebraic equations (Step 6) has been eliminated. Steps 3 to 5 serve to determine the functions $p^0(t)$, $u(t)$, and $x^1(t)$ uniquely. In other respects the solving procedure remains unaltered.

Let us consider the problem of minimizing the functional

$$J = \int_{t_0}^{T} f_*(x, u)\,dt, \qquad f_*(x, u) = f_*^0(x, u) + \varepsilon f_*^1(x, u) + \ldots$$

where f_* is a given function. The equations and boundary conditions take the form (1.1) and (1.2) as before; expansions (1.10) remain valid. If f_*^0 is independent of u, we introduce a new phase coordinate and a new functional by means of the relations

$$dx_* / dt = f_* = f_*^0(x) + \varepsilon f_*^1(x, u) + \ldots, \; x_*(t_0) = 0, \quad J_* = J = x_*(T)$$

On the other hand, if f_*^0 depends explicitly on u, we set

$$dx_* / dt = \varepsilon f_* = \varepsilon f_*^0(x, u) + \ldots, \; x_*(t_0) = 0, \quad J_* = \varepsilon J = x_*(T)$$

We increase by unity the dimensionality of the vector x by adding to it the new component x_*. The initial problem equivalent to that of minimizing the functional J_* then reduces to the case considered above (in Sections 1 and 2). By the procedure of Section 2 we can determine the minimum of the functional J_* to within an error on the order of ε^2. The error of the solution for the initial functional J is on the order of ε^2 if f_*^0 is independent of u, and on the order of ε if f_*^0 depends explicitly on u.

The above method can be applied to the construction of approximate analytic solutions of optimal control problems in the case of weakly controlled systems. Furthermore, the technique can be used to obtain an initial approximation for subsequent solution of the problem on a computer by various numerical methods, e.g. by the method described in [4]. In the latter case the parameter ε need not be very small.

It should be noted that problems of control of mechanical objects often involve the class of weakly controlled systems just considered. The parameter ε characterizes the ratio of the controlled forces (e.g. the thrust of the craft) to the uncontrolled forces (e.g. the weight).

The approach described (i.e. expansion in the small parameter) is also applicable to differential game problems provided the system is weakly controlled relative to one or both players.

3. Local optimality. Since the first integrals (2.5) of the zeroth-approximation system are assumed known, they can be taken as the new required functions in system (1.1). In other words, Eqs. (2.4) and (2.5) can be considered as direct and inverse transformations from the vector of variables x to the vector of new variables c; the vector c is considered constant in the zeroth approximation only. Such a transformation is often employed in celestial mechanics, where the variables of the c type are called "osculating elements".

Let us consider the solution of Section 2 choosing as our phase coordinates the first integrals of the zeroth-approximation system (i.e. the osculating variables c from (2.5)) and that these variables are denoted by x as before. The solving procedure of Section 2 then remains unchanged, although some simplifications occasioned by the choice of phase coordinates do arise.

Since the new phase coordinates are identically constant in the zeroth approximation we must set $f^0 \equiv 0$ in the relations of Section 2. Here, as we see, the functions ϕ, g from (2.4) and (2.5) and matrices (2.7) are given by

$$f^0 \equiv 0,\ \varphi(t,\ c) = c, \qquad g(x,\ t) = x,\ \Phi(t,\ c) = G(t,\ c) = E \qquad (3.1)$$

where E is a unit matrix. Relations (2.6), (2.8), (2.10), and (2.13) become

$$x^0(t) = a^0,\ x^1(t) = a^1 + \int_{t_0}^{t_1} f^1(a^0,\ \tau,\ u(\tau))\, d\tau$$

$$p^0(t) = s, \qquad (s,\ f^1(a^0,\ t,\ u)) \to \sup \text{ with respect to } u \in U \qquad (3.2)$$

The remaining equations of Section 2 can also be simplified by substituting in them relations (3.1) and (3.2).

Let us make two further assumptions. First, we assume that the boundary conditions $q = 0$ are lacking at the end of the process. As stated at the end of Section 2, this enables us to omit in the equations of Section 2 all terms containing $\lambda_i{}^0$ and $q_i{}^0$ and to simplify the solving procedure. Second, we assume that one of the two following conditions is fulfilled: either the function F^0 does not depend explicitly on t, or h^0 does not depend explicitly on x, i.e. the equation

$$(\partial F^0/\ \partial t)\ (\partial h^0\ /\ \partial x) = 0 \qquad (3.3)$$

is valid.

Condition (3.3) is fulfilled, for example, if $h(x,\ t) = t - T_*$, where T_* is a given number. Then the instant T of termination of the process defined by condition (1.2) is fixed and equal to T_*; moreover, $T^0 = T_*$, $T^1 = 0$.

Bearing in mind the above assumptions and Eqs. (3.1) to (3.3), we find λ^0 from relation (2.2) and then s from (2.11),

$$\lambda^0 = (\partial F^0/\partial t)\ (\partial h^0/\ \partial t)^{-1}, \quad s = -\ \partial F^0\ /\ \partial x \quad \text{for } x = a^0,\ t = T^0 \qquad (3.4)$$

Let us substitute Eq. (3.4) into the last condition of (3.2),

$$(s,\ f^1\ (a^0,\ t,\ u)) = -\ (\partial F^0\ /\ \partial x,\ f^1\ (a^0,\ t,\ u)) = \varepsilon^{-1}\ (\partial F^0\ /\ \partial t - dF^0\ /\ dt) \quad (3.5)$$

By virtue of Eq. (1.1), the total derivative here must be computed with allowance for terms of the first order of smallness, i.e. for $f = \varepsilon f^1$. Without reducing the accuracy of the solution (whose error is of a higher order of smallness), we can replace this derivative by the derivative given by exact Eqs. (1.1).

According to the last condition of (3.2) the approximate optimal control maximizes the left-hand expression of (3.5). Since the derivative $\partial F^0/\partial t$ does not depend explicitly on u, by virtue of Eqs. (3.5) the control can be determined from the condition of minimality of the total derivative dF^0/dt.

The control which at each instant minimizes the rate of change dF^0/dt of the functional F^0 being minimized is often called "locally optimal". Thus, we have just shown that in a weakly controlled system a locally optimal control is, under the above conditions, an approximately optimal control. In other words, the values of the functional for the exact optimal and locally optimal controls differ by a quantity on the order of ε^2.

Locally optimal controls are usually quite easy to find. It is sufficient to write out the total derivative dF^0/dt as a function of the osculating variables, the control, and time, and to find its minimum with respect to $u \in U$. The control is then obtained as a function of the osculating phase coordinates and possibly of time, i.e. in synthetic form. After this the trajectory can be determined either analytically (as in Sections 2 and 3) or by numerical solution of the Cauchy problem. Owing to their simplicity locally optimal controls have been used on many occasions in solving problems on controlled flights of low-thrust spacecraft (see the survey and bibliography in [5]). The role of the zeroth approximation is here played by the Keplerian motion; the ordinary osculating elements serve as the first integrals of the equations of the zeroth approximation. Locally optimal controls have also been used as initial approximations in numerical computations of optimal trajectories. The above results indicate under what conditions and in what sense locally optimal controls are, in fact, close to optimal controls.

4. The maximum gliding range problem. To illustrate the general approach described in Section 2 let us consider the following model problem solved numerically in [4]. In aircraft (material point) is in plane motion in the atmosphere. We denote its initial velocity by v_0, the constant acceleration due to gravity by g, and the mass of the craft by m; we take the quantities $l = v_0{}^2 g^{-1}$, $v_0 g^{-1}$, and m as our units of length, time, and mass, respectively. The relationships between the dimensional and dimensionless variables are as follows:

$$t^* = v_0 g^{-1} t, \quad x_i^* = l x_i, \quad x_j^* = v_0 x_j, \quad v^* = v_0 v \quad (i = 1, 2;\ j = 3, 4) \tag{4.1}$$

Here t is the time, x_1 the horizontal coordinate (range), x_2 the vertical coordinate (altitude), x_3 and x_4 the horizontal and vertical velocity components, and v the average value of the velocity; the asterisks denote the corresponding dimensional quantities. In addition to weight, the craft is acted upon by aerodynamic forces, i.e. by the drag R and the lift Y, which are given by

$$R = {}^1/_2\ \rho^*(v^*)^2 S^* C_x, \qquad Y = {}^1/_2 \rho^*(v^*)^2 S^* C_y \tag{4.2}$$

The drag R is directed opposite to the velocity of the craft; the lift Y is directed perpendicularly to it. Here ρ^* is the density of the atmosphere, S^* is the characteristic surface area of the craft, and C_x and C_y are aerodynamic coefficients which depend on the angle of attack α. Let the control be effected by varying the angle α and the surface area S^*, which can assume one of the two values $S_1{}^*$ and $S_2{}^*$, where $S_1{}^* < S_2{}^*$. The latter means of control qualitatively simulates a change in wing geometry or an extension of flaps.

Let us rewrite Eqs. (4.2), introducing the dimensionless variables

$$R = \varepsilon m g \rho v^2 S C_x, \qquad Y = \varepsilon m g \rho v^2 S C_y$$
$$\rho = \frac{\rho^*}{\rho_0{}^*}, \qquad S = \frac{S^*}{S_1{}^*}, \qquad \varepsilon = \frac{\rho_0{}^* v_0{}^2 S_1{}^*}{2mg} \tag{4.3}$$

Here $\rho_0{}^*$ is the density of the atmosphere at the initial altitude, ρ is the dimensionless density, and S is a dimensionless quantity which assumes the values $S_1 = 1$ and $S_2 = S_2{}^*/S_1^*$ > 1; the dimensionless parameter ε characterizes the ratio of the aerodynamic forces to the weight. Let us write out the equations of motion of the craft in dimensionless variables (4.1), projecting forces (4.3) on the axes x_1 and x_2,

$$\frac{dx_1}{dt} = x_3, \qquad \frac{dx_2}{dt} = x_4, \qquad \frac{dx_3}{dt} = -\varepsilon \rho v S\,(C_x x_3 + C_y x_4)$$
$$\frac{dx_4}{dt} = -1 + \varepsilon \rho v S\,(C_y x_3 - C_x x_4) \tag{4.4}$$

We specify the initial conditions in the form

$$x_1 = x_2 = 0, \quad x_3 = \cos\theta_0, \quad x_4 = \sin\theta_0 \quad \text{for } t = 0 \quad (0 < \theta_0 < \pi/2) \tag{4.5}$$

Here θ_0 is the given initial tilt angle of the trajectory (the initial velocity in the dimensionless variables is equal to unity). We pose the following variational problem: to achieve the maximum flight (gliding) range x_1 at the instant when the altitude x_2 is again equal to zero. The controlling functions are the angle of attack $\alpha(t)$ on which C_x and C_y depend (we shall define this dependence below) and the quantity $S(t)$ which assumes the discrete values S_1 and S_2. This problem conforms to the general formulation of Section 1 provided the parameter ε is small (which we in fact assume to be the case). In the notation of Section 1 we have

$$h^0 = x_2, \quad h^1 = 0, \quad F^0 = -x_1, \quad F^1 = 0$$

and the boundary conditions $q = 0$ of (1.2) are lacking. The functions f_k^0 and f_k^1 are equal to the coefficients of ε^0 and ε in the right sides of system (4.4). We shall now follow the general procedure of Section 2.

1. We set $\varepsilon = 0$ in Eqs. (4.4) and find the general solution of the zeroth-approximation system which describes the system in the absence of drag.

$$x_1 = c_3 t + c_1, \quad x_2 = c_4 t + c_2 - t^2 / 2, \quad x_3 = c_3, \quad x_4 = c_4 - t \tag{4.6}$$

The right sides of these equations are the functions ϕ_k of (2.4). Solving Eqs. (4.6) for the constants c_i we obtain the first integrals (2.5) of the zeroth-approximation system

$$g_1 = x_1 - x_3 t, \quad g_2 = x_2 - x_4 t - t^2/2, \quad g_3 = x_3, \quad g_4 = x_4 + t \tag{4.7}$$

Making use of Eqs. (4.6) and (4.7), we construct matrices (2.7),

$$\Phi = \begin{Vmatrix} 1 & 0 & t & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad G = \begin{Vmatrix} 1 & 0 & -t & 0 \\ 0 & 1 & 0 & -t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}$$

2. The phase coordinates in zeroth-approximation (2.6) can be found by determining the arbitrary constants in (4.6) with the aid of initial conditions (4.5). We obtain

$$x_1^0 = t\cos\theta_0, \quad x_2^0 = t\sin\theta_0 - t^2 / 2, \quad x_3^0 = \cos\theta_0, \quad x_4^0 = \sin\theta_0 - t \tag{4.8}$$

Substituting solution (4.8) into the condition of termination of the process $x_2 = 0$ and determining the time T^0, we obtain the minimized functional J^0 (which is in our case equal to the range taken with the minus sign),

$$T^0 = 2\sin\theta_0, \qquad J^0 = -x_1(T^0) = -\sin 2\theta_0$$

3. Substituting the resulting solution into general relations (2.2), (2.10), and (2.11), we obtain, in succession, λ^0, p^0, and s,

$$\lambda^0 = -x_3^0(T^0) / x_4^0(T^0) = \operatorname{ctg}\theta_0, \quad s_1 = 1, \quad s_2 = \operatorname{ctg}\theta_0, \; s_3 = T^0 = 2\sin\theta_0$$

$$s_4 = T^0 \operatorname{ctg}\theta_0 = 2\cos\theta_0, \quad p_1^0 = 1, \quad p_2^0 = \operatorname{ctg}\theta_0, \quad p_3^0 = T^0 - t$$

$$p_4^0 = \operatorname{ctg}\theta_0 (T^0 - t)$$

4. We now find from relation (2.12) that the controlling functions can be determined from the condition of maximality of the following expression with respect to α and S:

$$\varepsilon\rho v S (T^0 - t) \; [\operatorname{ctg}\theta_0 (C_y x_3^0 - C_x x_4^0) - (C_x x_3^0 + C_y x_4^0)]$$

Substituting solution (4.8) into this expression and recalling that $t \leqslant T^0 = 2\sin\theta_0$, we can rewrite the above condition as

$$S\left[C_x - C_y\left(\frac{\cos 2\theta_0 + t\sin\theta_0}{\sin 2\theta_0 - t\cos\theta_0}\right)\right] \to \min \quad \text{with respect to } \alpha, S \tag{4.9}$$

If no restrictions are imposed on the angle of attack α, then fulfillment of condition (4.9) requires that the first derivative of Expression (4.9) with respect to α equal zero. From this we find that

$$\frac{C_y'(\alpha)}{C_x'(\alpha)} = \frac{\sin 2\theta_0 - t\cos\theta_0}{\cos 2\theta_0 + t\sin\theta_0} \tag{4.10}$$

where the prime denotes the derivative with respect to α.

The second derivative of (4.9) with respect to α must be nonnegative. With the aid of Eq. (4.10) we can rewrite this condition as

$$C_x'' - (C_x' / C_y') C_y'' = C_y' (C_x' / C_y')' \geqslant 0 \tag{4.11}$$

Thus, the control $\alpha(t)$ can be determined from condition (4.9) by satisfying conditions (4.10) and (4.11). If conditions (4.10) and (4.11) determine α uniquely, then this α is the one required. Once α has been found, the control S can be chosen in accordance with the sign of the coefficient of S in (4.9). With allowance for Eq. (4.10), we can express the condition for choosing S in the form

$$S = S_1 \text{ for } A > 0, \quad S = S_2 > S_1 \text{ for } A < 0 \qquad A = C_x - C_y\,(C_x' / C_y') \tag{4.12}$$

Let us interpret condition (4.10) geometrically. Let $\theta(t)$ be the trajectory tilt angle with respect to the horizontal axis in the zeroth-approximation. By (4.8) we have

$$\operatorname{tg}\theta = x_4^0 / x_3^0 = (\sin\theta_0 - t) / \cos\theta_0 \tag{4.13}$$

It is not difficult to verify that Eq. (4.10) with allowance for (4.13) can be written as

$$C_y' / C_x' = \operatorname{tg}(\theta + \theta_0) \tag{4.14}$$

The functions $C_x(\alpha)$ and $C_y(\alpha)$ define parametrically the equation of the polar curve of the craft, i.e. the polar curve in the plane C_x, C_y. Eq. (4.14) shows that with an optimal angle of attack $\alpha(t)$ the tangent to the polar curve of the craft at any instant forms the angle $\theta + \theta_0$ with the axis C_x.

T make our computations specific let us take as our aerodynamic characteristics

$$C_x = 1 - \cos 2\alpha_0 \cos 2\alpha, \qquad C_y = K \sin 2\alpha_0 \sin 2\alpha \tag{4.15}$$

Here α_0 and K are constants. As we can readily verify, K is equal to the maximum lift/drag ratio (max(C_y/C_x); α_0 is the angle of attack for which this maximum is achieved. Relations (4.15) are those taken in [4]. They have the following properties typical of aircraft: (1) the functions C_x and C_y are periodic in α; (2) $C_x(\alpha)$ is an even, and $C_y(\alpha)$ an odd function of α, which is the case with symmetrical craft; (3) for small α the functions (4.15) have the usual form $C_x = C_1 + C_2\alpha^2$, $C_y = C_3\alpha$, where C_1, C_2, and C_3 are constants. The polar curve of a craft having characteristics (4.15) is an ellipse.

Substituting relations (4.15) into conditions (4.10) to (4.12), we obtain

$$\operatorname{tg} 2\alpha = K \operatorname{tg} 2\alpha_0 \frac{\cos 2\theta_0 + t \sin\theta_0}{\sin 2\theta_0 - t\cos\theta_0}, \qquad \frac{\cos 2\alpha_0}{\cos 2\alpha} \geqslant 0, \qquad A = 1 - \frac{\cos 2\alpha_0}{\cos 2\alpha} \tag{4.16}$$

To be specific, let us take $\alpha_0 < \pi/4$, $\theta_0 < \pi/4$ (other cases can be considered in the same way). Bearing in mind the inequality $t \leqslant T^0 = 2\sin\theta_0$, we find from the first Eq. of (4.16) that $\operatorname{tg} 2\alpha \geqslant 0$. Recalling the second relation of (4.16), we find that $0 \leqslant 2\alpha \leqslant \pi/2$. The angle α can be determined in the same way, and conditions (4.16) and (4.12) become

$$\alpha(t) = \frac{1}{2}\operatorname{arctg}\left(K \operatorname{tg} 2\alpha_0 \frac{\cos 2\theta_0 + t\sin\theta_0}{\sin 2\theta_0 - t\cos\theta_0}\right) \tag{4.17}$$

$$(S = S_1 \text{ for } \alpha < \alpha_0, \; S = S_2 \text{ при } \alpha > \alpha_0)$$

Thus, the controlling functions have been determined completely. According to (4.17) the angle $\alpha(t)$ increases monotonously from $\alpha(0)$ to $\pi/4$. The piecewise-continuous function $S(t)$ clearly changes value (switches over) not more than once. At the end of the process, since $\alpha_0 < \pi/4$, it assumes its larger value S_2. If $\alpha(0) \geqslant \alpha_0$ there is no switchover and $S = S_2$ everywhere; if $\alpha(0) < \alpha_0$, then $S = S_1$ over the initial portion of the trajectory. The larger lift/drag ratio K, the earlier the switchover and the larger the value assumed by the angle α at the same instant.

The above results are in good agreement with those of [4], where the exact solution of the problem for the case of constant atmospheric density is obtained over a wide range of values (from 0.1 to 3) of the parameters ε and K. Comparison of the optimal control law $\alpha(t)$ from [4] for $\alpha_0 = \theta_0 = 10°$ with law (4.17) indicates that for $\varepsilon = 0.1$ the two laws practically coincide; even for $\varepsilon = 0.5$ the difference between them is about 10% over the entire range of K values. The instant of switchover defined by condition (4.17) is also in good agreement with the computed results (within approximately the same error margin).

Approximate analytic solution (4.17) was obtained for an arbitrary dependence of atmospheric density on altitude. Specifying this dependence, we can readily use quadrature (2.8) to find the correction for the trajectory due to aerodynamic forces. We note that the trajectory and functional are then determined to within an error on the order of ε^2, i.e. by one order of ε more accurately than the control.

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